Problem 2.14

Use the method of Problem 2.7 to solve the following: A mass m is constrained to move along the x axis subject to a force $F(v) = -F_{\text{o}}e^{v/V}$, where F_{o} and V are constants. **(a)** Find $v(t)$ if the initial velocity is $v_{\text{o}} > 0$ at time $t = 0$. **(b)** At what time does it come instantaneously to rest? **(c)** By integrating $v(t)$, you can find $x(t)$. Do this and find how far the mass travels before coming instantaneously to rest.

Solution

Part (a)

Consider a mass constrained to move on the x-axis. By Newton's second law,

$$F = ma.$$

If the force is $F(v) = -F_{\text{o}}e^{v/V}$, then

$$-F_{\text{o}}e^{v/V} = ma$$

$$-F_{\text{o}}e^{v/V} = m\frac{dv}{dt}.$$

To solve for v, separate variables

$$-\frac{F_{\text{o}}}{m}\,dt = e^{-v/V}\,dv$$

and integrate both sides.

$$\int -\frac{F_{\text{o}}}{m}\,dt = \int e^{-v/V}\,dv$$

$$-\frac{F_{\text{o}}}{m}t + C = -Ve^{-v/V}$$

Apply the prescribed initial condition $v(0) = v_{\text{o}}$ to determine C.

$$C = -Ve^{-v_{\text{o}}/V}$$

As a result,

$$-Ve^{-v/V} = -\frac{F_{\text{o}}}{m}t - Ve^{-v_{\text{o}}/V}$$

$$e^{-v/V} = \frac{F_{\text{o}}}{mV}t + e^{-v_{\text{o}}/V}$$

$$\ln e^{-v/V} = \ln\left(\frac{F_{\text{o}}}{mV}t + e^{-v_{\text{o}}/V}\right)$$

$$-\frac{v}{V} = \ln\left(\frac{F_{\text{o}}}{mV}t + e^{-v_{\text{o}}/V}\right).$$

Therefore, the velocity is

$$v(t) = -V \ln\left(\frac{F_o}{mV}t + e^{-v_o/V}\right).$$

Part (b)

To find the time it takes for the object to come to rest, set $v(t_{\text{rest}}) = 0$ and solve the equation for t_{rest}.

$$0 = -V \ln\left(\frac{F_o}{mV}t_{\text{rest}} + e^{-v_o/V}\right)$$

$$0 = \ln\left(\frac{F_o}{mV}t_{\text{rest}} + e^{-v_o/V}\right)$$

This means that

$$\frac{F_o}{mV}t_{\text{rest}} + e^{-v_o/V} = 1.$$

Solve for the term with t_{rest}.

$$\frac{F_o}{mV}t_{\text{rest}} = 1 - e^{-v_o/V}$$

Therefore,

$$t_{\text{rest}} = \frac{mV}{F_o}\left(1 - e^{-v_o/V}\right).$$

Part (c)

Replace $v(t)$ with dx/dt in the result of part (a).

$$\frac{dx}{dt} = -V \ln\left(\frac{F_o}{mV}t + e^{-v_o/V}\right)$$

Solve for x by separating variables again.

$$dx = -V \ln\left(\frac{F_o}{mV}t + e^{-v_o/V}\right) dt$$

Integrate both sides definitely, assuming the position at $t = 0$ is x_o.

$$\int_{x_o}^{x} dx' = \int_0^t -V \ln\left(\frac{F_o}{mV}t' + e^{-v_o/V}\right) dt'$$

$$x - x_o = -V \int_0^t \ln\left(\frac{F_o}{mV}t' + e^{-v_o/V}\right) dt'$$

Make a substitution.

$$u = \frac{F_o}{mV}t' + e^{-v_o/V}$$

$$du = \frac{F_o}{mV}\,dt' \quad \rightarrow \quad dt' = \frac{mV}{F_o}\,du$$

Consequently,

$$\begin{aligned}
x(t) &= x_o - V\int_{e^{-v_o/V}}^{\frac{F_o}{mV}t+e^{-v_o/V}} \ln u \left(\frac{mV}{F_o}\,du\right) \\
&= x_o - \frac{mV^2}{F_o}\int_{e^{-v_o/V}}^{\frac{F_o}{mV}t+e^{-v_o/V}} \ln u\,du \\
&= x_o - \frac{mV^2}{F_o} u(\ln u - 1)\Big|_{e^{-v_o/V}}^{\frac{F_o}{mV}t+e^{-v_o/V}} \\
&= x_o - \frac{mV^2}{F_o}\left\{\left(\frac{F_o}{mV}t + e^{-v_o/V}\right)\left[\ln\left(\frac{F_o}{mV}t + e^{-v_o/V}\right) - 1\right] - e^{-v_o/V}\left(\ln e^{-v_o/V} - 1\right)\right\} \\
&= x_o + \frac{mV^2}{F_o}\left(\frac{F_o}{mV}t + e^{-v_o/V}\right)\left[1 - \ln\left(\frac{F_o}{mV}t + e^{-v_o/V}\right)\right] + \frac{mV^2}{F_o}e^{-v_o/V}\left(-\frac{v_o}{V} - 1\right) \\
&= x_o + Vt - \frac{mVv_o}{F_o}e^{-v_o/V} - \frac{mV^2}{F_o}\left(\frac{F_o}{mV}t + e^{-v_o/V}\right)\ln\left(\frac{F_o}{mV}t + e^{-v_o/V}\right).
\end{aligned}$$

Therefore, the distance the mass travels before coming to rest is

$$\begin{aligned}
\Delta x &= x(t_{\text{rest}}) - x_o \\
&= Vt_{\text{rest}} - \frac{mVv_o}{F_o}e^{-v_o/V} - \frac{mV^2}{F_o}\left(\frac{F_o}{mV}t_{\text{rest}} + e^{-v_o/V}\right)\ln\left(\frac{F_o}{mV}t_{\text{rest}} + e^{-v_o/V}\right) \\
&= V\frac{mV}{F_o}\left(1 - e^{-v_o/V}\right) - \frac{mVv_o}{F_o}e^{-v_o/V} - \frac{mV^2}{F_o}\left(1 - e^{-v_o/V} + e^{-v_o/V}\right)\ln\left(1 - e^{-v_o/V} + e^{-v_o/V}\right) \\
&= \frac{mV^2}{F_o}\left(1 - e^{-v_o/V}\right) - \frac{mVv_o}{F_o}e^{-v_o/V} - \frac{mV^2}{F_o}(1)\ln 1 \\
&= \frac{mV^2}{F_o}\left(1 - e^{-v_o/V}\right) - \frac{mVv_o}{F_o}e^{-v_o/V} \\
&= \frac{mV^2}{F_o}\left[\left(1 - e^{-v_o/V}\right) - \frac{v_o}{V}e^{-v_o/V}\right] \\
&= \frac{mV^2}{F_o}\left[1 - \left(1 + \frac{v_o}{V}\right)e^{-v_o/V}\right].
\end{aligned}$$